DETERMINATION OF THE STEADY TEMPERATURE FIELD IN A TOOTHED RING

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An examination is made of a method of calculating the temperature field in the region illustrated in Fig. 1. The temperature T_0 of the shaded heat regions is found, under the assumption that the shaded regions possess high thermal conductivity, and that the temperature is the same at all points of these regions. The method is applied to calculation of the temperature field in a phase shifter rotor. The dependence of T_0 on the thickness of thermal insulation of the rotor conductors is given.

A method is described for determining the temperature of heat sources located in notches in an annular region and separated from it by a layer of thermally insulating material. The temperature field in the annular region is constructed by joining solutions of the heat conduction equation.

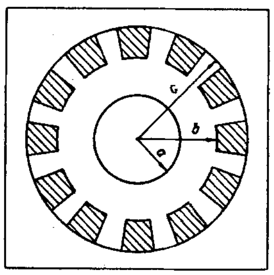

Fig. 1. Region in which the temperature field is determined.

The shaded curved rectangles of Fig. 1 are sections of the material which possess much larger thermal conductivity in comparison with the material of the remaining part of the annulus. These sections are surrounded by a thin layer of thermally insulating material. In the part of the annulus where $a \le r \le b$, there is heat generation with a volume density Q_1; in the remaining part of the annulus, included between the shaded rectangles, the density of heat generation is Q_2. We shall denote by q the total amount of heat liberated in a single shaded rectangle in unit time. The heat flux through the boundary $r = c$ is equal to zero, and the temperature of the annulus is zero on the inner boundary at $r = a$, i.e.,

$$\left.\frac{\partial T}{\partial r}\right|_{r=c} = 0, \tag{1}$$

$$T|_{r=a} = 0. \tag{2}$$

Because the thermal conductivity of the material of the shaded sections is considerably greater than that of the remaining part of the annulus, while the size of these sections is not large, we may consider the temperature at all points of these sections to be the same. We denote it by T_0.

Thus, the problem reduces to solution of the Poisson equation for the region shown in Fig. 2, under the following boundary conditions:

$$\left.\frac{\partial T}{\partial \varphi}\right|_{\substack{\varphi=0\\ \varphi=\alpha_2}} = 0, \quad \left.\frac{\partial T}{\partial r}\right|_{r=c} = 0, \quad T|_{r=a} = 0; \tag{3}$$

$$k_1 \left.\frac{\partial T}{\partial r}\right|_{r=b} = \frac{k}{\delta}(T_0 - T)|_{r=b}, \qquad \alpha_1 \leqslant \alpha \leqslant \alpha_2; \tag{4}$$

$$k_1 \left.\frac{\partial T}{r\partial \varphi}\right|_{\varphi=\alpha_1} = \frac{k}{\delta}(T_0 - T)|_{\varphi=\alpha_1}, \qquad b \leqslant r \leqslant c. \tag{5}$$

The unknown temperature T_0 appears in the boundary conditions (4) and (5). Its determination can be a matter of very great interest, since it is evident that it will exceed the temperature at any point of the annulus. To determine T_0 we proceed as follows. We assume that an expression for the temperature at any point of the region shown in Fig. 2 has been constructed. T_0 will appear in this expression as a parameter. The expression constructed permits us to calculate the heat flux through the boundary $r = a$, but it is determined by the quantities Q_1, Q_2, q. This makes it possible to find T_0.

We shall calculate the temperature field in regions I and II (see Fig. 2). Following the method of Grinberg [1], we note, that the eigenfunctions in φ for regions I and II do not coincide:

$$\Phi_{In}(\varphi) = \cos \lambda'_n \varphi, \quad \lambda'_n = n\pi\alpha_2; \tag{6}$$

$$\Phi_{IIn}(\varphi) = \cos \lambda_n \varphi, \tag{7}$$

where

$$\lambda_n = \gamma_n / \alpha_1; \quad \mathrm{tg}\,\gamma_n = hb\alpha_1 \gamma_n; \quad h = k k_1 \delta. \tag{8}$$

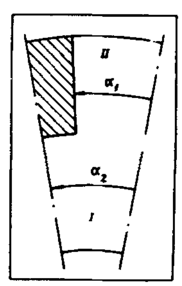

Fig. 2. Region for which the boundary problem is solved.

Therefore, to join the solutions we use the following means. Since $T_{II}|_{r=b}$ and $\left.\frac{\partial T_I}{\partial r}\right|_{r=b}$ we may consider the φ to be arbitrary, and expand them according to a complete system of eigenfunctions (7) and (6), respectively:

$$T_{\mathrm{II}}\big|_{r=b}=\frac{4}{\alpha_1}\sum_{m=1}^{\infty}\frac{\gamma_m}{2\gamma_m+\sin 2\gamma_m}G_m^{(2)}\cos\lambda_m\varphi; \qquad (9)$$

$$\left.\frac{\partial T_{\mathrm{I}}}{\partial r}\right|_{r=b}=\sum_{m=1}^{\infty}G_m^{(1)}\cos\lambda'_m\varphi. \qquad (10)$$

Here the factors $4\gamma_m/\alpha_1(2\gamma_m+\sin 2\gamma_m)$ have been separated from the arbitrary coefficients $G_m^{(2)}$, for convenience.

Expressing the constants of integration of the Poisson equation for regions II and I in terms of the constants of the expansions (9) and (10), we have

$$T_{\mathrm{II}}(r,\varphi)=$$

$$=\frac{4}{\alpha_1}\sum_{n=1}^{\infty}\frac{\gamma_n}{2\gamma_n+\sin 2\gamma_n}\cos\gamma_n\frac{\varphi}{\alpha_1}\left\{\frac{bhT_0}{\lambda_n^2}\cos\gamma_n+\right.$$

$$+\frac{Q_2\sin\gamma_n}{k\lambda_n(\lambda_n^2-4)}r^2+\frac{2Q_2c^2\sin\gamma_n}{k\lambda_n^2(4-\lambda_n^2)}\left(\frac{r}{c}\right)^{\lambda_n}+$$

$$+\left[G_n^{(2)}-\frac{Q_2b^2\sin\gamma_n}{k\lambda_n(\lambda_n^2-4)}-\frac{bhT_0\cos\gamma_n}{\lambda_n^2}+\right.$$

$$\left.\left.+\frac{2Q_2c^2(b/c)^{\lambda_n}\sin\gamma_n}{k\lambda_n^2(\lambda_n^2-4)}\right]\frac{(c/r)^{\lambda_n}+(r/c)^{\lambda_n}}{(b/c)^{\lambda_n}+(c/b)^{\lambda_n}}\right\}; \qquad (11)$$

$$T_{\mathrm{I}}(r,\varphi)=-\frac{Q_1}{4k}(a^2-r^2)+\frac{b\ln(r/a)}{1+bh\ln(b/a)}S_0+$$

$$+2\sum_{n=1}^{\infty}S_n\frac{(r/a)^{\lambda'_n}-(a/r)^{\lambda'_n}}{(h+\lambda'_n/b)(b/a)^{\lambda'_n}-(h-\lambda'_n/b)(a/b)^{\lambda'_n}}\cos\lambda'_n\varphi. \qquad (12)$$

where

$$S_0=\frac{Q_1b}{2k}+\frac{Q_1h(b^2-a^2)}{4k}+$$

$$+hT_0\left(1-\frac{\alpha_1}{\alpha_2}\right)+\sum_{m=0}^{\infty}G_m^{(1)}\frac{1}{m\pi}\sin\frac{m\pi\alpha_1}{\alpha_2}+$$

$$+\frac{4}{\alpha_2}h\sum_{m=1}^{\infty}\frac{\sin\gamma_m}{2\gamma_m+\sin 2\gamma_m}G_m^{(2)}; \qquad (13)$$

$$S_n=\sum_{m=0}^{\infty}G_m^{(1)}\left[\sin(m-n)\pi\frac{\alpha_1}{\alpha_2}\Big/2(m-n)\pi+\right.$$

$$\left.+\sin(m+n)\pi\frac{\alpha_1}{\alpha_2}\Big/2(m+n)\pi\right]+$$

$$+\frac{2hT_0}{n\pi}\cos\frac{n\pi(\alpha_1+\alpha_2)}{\alpha_2}\sin\frac{n\pi(\alpha_2-\alpha_1)}{\alpha_2}+$$

$$+\frac{2h}{\alpha_2}\sum_{m=1}^{\infty}G_m^{(2)}\frac{\gamma_m}{2\gamma_m+\sin 2\gamma_m}\times$$

$$\times\left[\sin\left(n\pi\frac{\alpha_1}{\alpha_2}-\gamma_m\right)\Big/\left(n\pi\frac{\alpha_1}{\alpha_2}-\gamma_m\right)+\right.$$

$$\left.+\sin\left(n\pi\frac{\alpha_1}{\alpha_2}+\gamma_m\right)\Big/\left(n\pi\frac{\alpha_1}{\alpha_2}+\gamma_m\right)\right]. \qquad (14)$$

The joining condition for $T_{\mathrm{I}}(r,\varphi)$ and $T_{\mathrm{II}}(r,\varphi)$ make it possible to express S_0 and S_n only in terms of the set of coefficients $G_m^{(2)}$, where $m = 1, 2, \ldots,$ and to make up an infinite system of linear algebraic equations for $G_m^{(2)}$:

$$S_0=\frac{1+bh\ln(b/a)}{\alpha_2 b\ln(b/a)}\times$$

$$\times\left[4\sum_{l=1}^{\infty}\frac{\sin\gamma_l\,\alpha_2/\alpha_1}{2\gamma_l+\sin 2\gamma_l}G_l^{(2)}-\frac{Q_1\alpha_2}{4k}(a^2-b^2)\right]; \qquad (15)$$

$$S_n=\frac{4}{\alpha_1\alpha_2}\frac{(h+\lambda'_n/b)(b/a)^{\lambda'_n}-(h-\lambda'_n/b)(a/b)^{\lambda'_n}}{(b/a)^{\lambda'_n}-(a/b)^{\lambda'_n}}\times$$

$$\times\sum_{l=1}^{\infty}G_l^{(2)}\frac{\gamma_l}{2\gamma_l+\sin 2\gamma_l}\left[\frac{\sin(n\pi/\alpha_2-\gamma_l/\alpha_1)\alpha_2}{2(n\pi/\alpha_2-\gamma_l/\alpha_1)}+\right.$$

$$\left.+\frac{\sin(n\pi/\alpha_2+\gamma_l/\alpha_1)\alpha_2}{2(n\pi/\alpha_2+\gamma_l/\alpha_1)}\right]; \qquad (16)$$

$$G_n^{(2)}=P_n+\sum_{m=1}^{\infty}D_{nm}G_m^{(2)}, \qquad (17)$$

where

$$n=1,2,\ldots;$$

$$P_n=\frac{bhT_0\cos\gamma_n}{\lambda_n^2}+$$

$$+\frac{Q_2b^2\sin\gamma_n}{k\lambda_n^2(\lambda_n^2-4)}\frac{(\lambda_n-2)(b/c)^{\lambda_n}-(\lambda_n+2)(c/b)^{\lambda_n}}{(b/c)^{\lambda_n}-(c/b)^{\lambda_n}}+$$

$$+\frac{4Q_2c^2\sin\gamma_n}{k\lambda_n^2(\lambda_n^2-4)}\frac{1}{(b/c)^{\lambda_n}-(c/b)^{\lambda_n}}-$$

$$-\frac{Q_1b^2\sin\gamma_n}{2k\lambda_n^2}\frac{(b/c)^{\lambda_n}+(c/b)^{\lambda_n}}{(b/c)^{\lambda_n}-(c/b)^{\lambda_n}}-$$

$$-\frac{Q_1}{4k}(a^2-b^2)\frac{\sin\gamma_n}{\lambda_n^2\ln(b/a)}\frac{(b/c)^{\lambda_n}+(c/b)^{\lambda_n}}{(b/c)^{\lambda_n}-(c/b)^{\lambda_n}}; \qquad (18)$$

$$D_{nm}=\frac{4\sin\gamma_n}{\lambda_n^2\alpha_2\ln(b/a)}\frac{\sin\gamma_m\,\alpha_2/\alpha_1}{2\gamma_m+\sin 2\gamma_m}\frac{(b/c)^{\lambda_n}+(c/b)^{\lambda_n}}{(b/c)^{\lambda_n}-(c/b)^{\lambda_n}}+$$

$$+\frac{8}{\alpha_1\alpha_2\gamma_n}\frac{(b/c)^{\lambda_n}+(c/b)^{\lambda_n}}{(b/c)^{\lambda_n}-(c/b)^{\lambda_n}}\sum_{l=1}^{\infty}\left[\frac{\sin(\gamma_n/\alpha_1-l\pi/\alpha_2)\alpha_1}{2(\gamma_n/\alpha_1-l\pi/\alpha_2)}+\right.$$

$$\left.+\frac{\sin(\gamma_n/\alpha_1+l\pi/\alpha_2)\alpha_1}{2(\gamma_n/\alpha_1+l\pi/\alpha_2)}\right]\left[\frac{\sin(l\pi/\alpha_2-\gamma_m/\alpha_1)\alpha_2}{2(l\pi/\alpha_2-\gamma_m/\alpha_1)}+\right.$$

$$\left.+\frac{\sin(l\pi/\alpha_2+\gamma_m/\alpha_1)\alpha_2}{2(l\pi/\alpha_2+\gamma_m/\alpha_1)}\right]\times$$

$$\times\frac{(b/a)^{\lambda'_l}+(a/b)^{\lambda'_l}}{(b/a)^{\lambda'_l}-(a/b)^{\lambda'_l}}\frac{\lambda'_l\gamma_m}{2\gamma_m+\sin 2\gamma_m}. \qquad (19)$$

It may be seen from (18) and from (19) that with increase of n, and therefore, of λ_n also, the moduli of the quantities P_n and D_{nm} decrease quite rapidly. Therefore, in the system (17), we may restrict ourselves to a small value of n.

By finding a finite number of coefficients $G_n^{(2)}$, and substituting them into formulas (11), (15) and (16), we obtain the temperature distribution in regions I and II.

By specifying the heat flux at $r = a$, we can find the temperature in the shaded region.

The method described may be applied conveniently for calculation of the temperature in the conductors of an electric motor rotor. We shall obtain the dependence of the temperature on the thickness of the thermal insulation of the conductors.

To calculate the temperature field in the rotor we use the following data: a = 12 cm; α_2 = 0.0872; Q_2 = 0.0221 W/cm^3; b = 24.5 cm; α_1 = 0.0548; Q_1 = 0.0331 W/cm^3; c = 29 cm; k = 0.221 W/cm^3; h = 0.007/δ cm, where δ is the thermal insulation thickness.

The heat flux through the boundary $r = a$ is equal to 2.52 W/cm. Then it follows from formulas (18) and (19), that P_2 and D_{2m} compose ~ 0.3% of P_1 and D_{1m} (m = 1.2). Therefore, restricting ourselves to this kind of accuracy, we take n = 1.

We find the constant of the expansion for

$$G_1^{(2)} = P_1/(1 - D_{11}), \tag{20}$$

$$P_1 = A_1 T_0 + B_1, \tag{21}$$

where A_1 B_1 and D_{11} are determined from Eqs. (18) and (19) with n = 1 and m = 1.

Making use of our knowledge of the total heat flux through the boundary $r = a$, we obtain

$$T_0 = \frac{1}{4k}\left[q \ln \frac{b}{a} + \frac{Qa^2\alpha_2}{4}\left(1 - \frac{b^2}{a^2} - 2 \ln \frac{b}{a}\right)\right] \times \\ \times \frac{2\gamma_1 + \sin 2\gamma_1}{\sin\gamma_1 \frac{\alpha_2}{\alpha_1}} \frac{1 - D_{11}}{A_1} - \frac{B_1}{A_1}. \tag{22}$$

By examining different thicknesses of thermal insulation, we find the dependence of T_0 on δ:

δ, cm	0.1	0.2	0.3	0.4	0.6
γ_1	0.297	0.210	0.174	0.149	0.123
T_0, °C	116	132	141	156	165

NOTATION

T_0 is the temperature of shaded region; a, b, c) are the dimensions shown in Fig. 1; α_1, α_2 are the angles shown in Fig. 2; T_I is the temperature in region I; T_{II} is the temperature in region II; k_1 is the thermal conductivity in regions I and II; k is the thermal conductivity of insulation; Q_1 is the volume heat release in region I; Q_2 is the volume heat release in region II; δ is the thickness of thermal insulation.

REFERENCES

1. G. A. Grinberg, Collection in honor of the 70-th birthday of Academician A. F. Ioffe [in Russian], Izd. AN SSSR, 1950.

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